The Beta Function & ODE-Convertible Integrals

ODE–Integration Bee Seminar Series

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Solutions to Monday's Homework

We will briefly prove both identities (using differentiation techniques) in today's session.

Homework 1

$$\int_0^\infty \frac{e^{-x} - e^{-ax}}{x}\,\mathrm{d}x = \log a, \qquad \Re(a) > 0.$$

Homework 2

$$\int_0^\infty \frac{\cos(2x)}{x^2 + 3^2}\,\mathrm{d}x = \frac{\pi}{6} e^{-6}.$$

Wallis and the Quarter Circle

Area of a Quarter Circle

Consider

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x,$$

the area of a quarter circle of radius 1.

Using the formula Area $= \dfrac{\pi r^2}{4}$ with $r = 1$, we know

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x = \frac{\pi}{4}.$$

“

Over seventy years before Euler, Wallis (1656) tried to compute the quarter-circle integral to reach a formula for π; but since he could only handle integrals of the form $\int_0^1 x^p(1-x)^q\,\mathrm{d}x$ (with p, q integers, or $q = 0$ and p rational). He used the value of the quarter-circle integral and some audacious guesswork to propose what became Wallis's product.

— George Andrews, Richard Askey, Ranjan Roy

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From the Quarter Circle to Cosine Powers

Proof Sketch: Substitution $x = \sin\theta$

Let

$$x = \sin\theta, \qquad \mathrm{d}x = \cos\theta\,\mathrm{d}\theta.$$

Then

$$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \cos\theta,$$

and as x moves from 0 to 1, the angle θ moves from 0 to $\frac{\pi}{2}$.

So

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x = \int_0^{\pi/2} \cos\theta \cdot \cos\theta\,\mathrm{d}\theta = \int_0^{\pi/2} \cos^2\theta\,\mathrm{d}\theta.$$

Thus

$$\int_0^{\pi/2} \cos^2\theta\,\mathrm{d}\theta = \frac{\pi}{4}.$$

The Quarter Circle of Radius 1

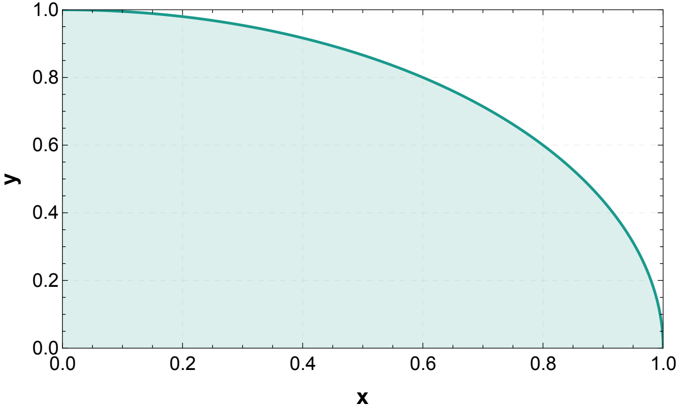

Figure: Graph of $y = \sqrt{1 - x^2}$ on $[0, 1]$

Wallis's Family of Integrals

Concept: Two Symmetric Families

For $n = 0, 1, 2, \ldots$ consider

$$I_n = \int_0^{\pi/2} \sin^n \theta \, d\theta, \qquad J_n = \int_0^{\pi/2} \cos^n \theta \, d\theta.$$

By symmetry of sine and cosine on $\left[0, \frac{\pi}{2}\right]$,

$$I_n = J_n \quad \text{for all } n.$$

We can just work with I_n, and remember that $I_2 = J_2 = \dfrac{\pi}{4}$.

Example: First Values

$$I_0 = \int_0^{\pi/2} 1 \, d\theta = \frac{\pi}{2}, \qquad I_1 = \int_0^{\pi/2} \sin \theta \, d\theta = 1.$$

The Recurrence for I_n

Proof Sketch: Integration by Parts

For $I_n = \int_0^{\pi/2} \sin^n \theta \, \mathrm{d}\theta$ and $n \geq 2$,

$$I_n = \int_0^{\pi/2} \sin^{n-1} \theta \, \sin \theta \, \mathrm{d}\theta.$$

Let

$$u = \sin^{n-1} \theta, \qquad \mathrm{d}v = \sin \theta \, \mathrm{d}\theta.$$

Then

$$\mathrm{d}u = (n-1) \sin^{n-2} \theta \cos \theta \, \mathrm{d}\theta, \qquad v = -\cos \theta.$$

Integration by parts gives

$$I_n = (n-1) \int_0^{\pi/2} \sin^{n-2} \theta \, \cos^2 \theta \, \mathrm{d}\theta.$$

Even and Odd Terms

From

$$I_n = \frac{n-1}{n} I_{n-2},$$

we get two chains:

- For even $n = 2k$:

$$I_{2k} = \frac{2k-1}{2k} I_{2k-2} = \frac{2k-1}{2k} \cdot \frac{2k-3}{2k-2} \cdots \frac{3}{4} \cdot \frac{1}{2} I_0.$$

- For odd $n = 2k+1$:

$$I_{2k+1} = \frac{2k}{2k+1} I_{2k-1} = \frac{2k}{2k+1} \cdot \frac{2k-2}{2k-1} \cdots \frac{2}{3} \cdot I_1.$$

So every I_n is a product of simple rational factors times either $I_0 = \frac{\pi}{2}$ or $I_1 = 1$.

An Inequality for I_{2n} and I_{2n+1}

For $0 \le \theta \le \frac{\pi}{2}$ we have

$$0 \le \sin\theta \le 1.$$

Fix an integer $n \ge 1$. Then

$$\sin^{2n+1}\theta \le \sin^{2n}\theta \le \sin^{2n-1}\theta,$$

because we are multiplying by another factor of $\sin\theta \in [0,1]$ each time.

Integrating over $[0, \frac{\pi}{2}]$ gives

$$I_{2n+1} \;\le\; I_{2n} \;\le\; I_{2n-1}.$$

Dividing by I_{2n+1}:

$$1 \le \frac{I_{2n}}{I_{2n+1}} \le \frac{I_{2n-1}}{I_{2n+1}} = \frac{2n+1}{2n},$$

where the equality uses the recurrence.

Applying the Squeeze Theorem

Concept: Limit of the Ratio

$$1 \le \frac{I_{2n}}{I_{2n+1}} \le \frac{2n+1}{2n}.$$

As $n \to \infty$,

$$\frac{2n+1}{2n} \to 1,$$

so the squeeze theorem gives

$$\boxed{\lim_{n\to\infty} \frac{I_{2n}}{I_{2n+1}} = 1.}$$

Constructing Wallis's Product

Using the explicit formulas:

$$I_{2n} = \prod_{k=1}^{n} \frac{2k-1}{2k} \cdot \frac{\pi}{2},$$

$$I_{2n+1} = \prod_{k=1}^{n} \frac{2k}{2k+1},$$

and the fact that

$$\frac{I_{2n}}{I_{2n+1}} \to 1,$$

we get

Wallis Product

$$\frac{\pi}{2} = \prod_{k=1}^{\infty} \frac{2k}{2k-1} \cdot \frac{2k}{2k+1}.$$

Summary of the Argument

Concept: Main Steps

- Define $I_m = \int_0^{\pi/2} \sin^m x \, \mathrm{d}x$.
- Derive recurrences for I_{2n} and I_{2n+1}.
- Use the inequality $\sin^{2n+1} x \leq \sin^{2n} x \leq \sin^{2n-1} x$.
- Apply the squeeze theorem to get $\frac{I_{2n}}{I_{2n+1}} \to 1$.
- Combine explicit products for I_{2n} and I_{2n+1}.

This yields Wallis's classical product for π.

Observation: A Useful Substitution

Under the substitution $t = x^2$, the quarter-circle integral becomes

$$\int_0^1 \sqrt{1 - x^2} \, \mathrm{d}x = \frac{1}{2} \int_0^1 t^{-1/2} (1 - t)^{1/2} \, \mathrm{d}t.$$

“

Of course, Wallis did not write his product as a limit or use the gamma function. Still, his result may have led Euler to consider the relation between the gamma function and integrals of the form $\int_0^1 x^p(1-x)^q \,\mathrm{d}x$, where p and q are not necessarily integers.

— George Andrews, Richard Askey, Ranjan Roy

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First Encounter

Concept: Guiding Idea

Many complicated integrals can be turned into a small number of *template* integrals by substitutions or parameter tricks. One of the cleanest templates is the Beta integral.

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t \qquad (x, y > 0).$$

- It depends on two real parameters x and y.
- The integrand is a simple product of powers on the fixed interval $[0, 1]$.
- In many problems, a messy integral becomes this shape after "good substitution(s)."

Goal: recognise when a complicated integral is just a Beta integral in disguise

Basic Identities We Will Use

Concept: Quick Checks on $B(x, y)$

Before using $B(x, y)$ as a template, we record a few identities that are easy to verify and will be used repeatedly.

- **Symmetry:**

$$B(x, y) \;=\; B(y, x),$$

  obtained from the substitution $t \mapsto 1 - t$ in the defining integral.

- **Two simple special cases:**

$$B(1, y) = \int_0^1 (1-t)^{y-1}\,\mathrm{d}t = \frac{1}{y}, \qquad B(x, 1) = \frac{1}{x}.$$

- **A structural relation (to remember for later):**

$$B(x, y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}.$$

Recognising the Beta Shape (I)

Concept: The Target Form

$$\int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t.$$

This form is built entirely from powers of t and $(1-t)$. Many integrals can be rearranged into this shape with a suitable substitution.

- The interval is fixed: 0 to 1.
- Only exponents change; the structure stays simple.
- Once the integrand matches this pattern, the result becomes $B(x, y)$.

Recognising the Beta Shape (II)

Concept: Common Routes to $(0, 1)$

Many integrals become Beta-shaped after a standard change of variable.

- **From** $(0, \infty)$**:**

$$t = \frac{x}{1 + x}.$$

  This substitution instantly produces factors of t and $(1 - t)$.

- **From** $[0, \frac{\pi}{2}]$**:**

$$u = \sin^2 \theta,$$

  which is useful for integrals involving powers of sin and cos.

- **Already on** $[0, 1]$**:** Aim to rewrite the integrand so that powers of t and $(1 - t)$ become visible.

Example: $(0,\infty)$ to $(0,1)$

Consider

$$\int_0^\infty \frac{x^{\alpha-1}}{(1+x)^{\alpha+\beta}}\,\mathrm{d}x.$$

Use the substitution

$$t = \frac{x}{1+x} \qquad \left(x = \frac{t}{1-t}\right).$$

Then

$$\mathrm{d}x = \frac{dt}{(1-t)^2}, \qquad 1+x = \frac{1}{1-t}, \qquad x^{\alpha-1} = \left(\frac{t}{1-t}\right)^{\alpha-1}.$$

After simplification:

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t \;=\; B(\alpha,\beta).$$

Reverse: $(0, 1)$ to $(0, \infty)$

Start with the Beta integral

$$\int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t.$$

Use the substitution

$$t = \frac{s}{1+s}$$

Then

$$dt = \frac{ds}{(1+s)^2}, \qquad t^{x-1} = \left(\frac{s}{1+s}\right)^{x-1}, \qquad (1-t)^{y-1} = (1+s)^{1-y}.$$

After simplification:

$$\int_0^\infty \frac{s^{x-1}}{(1+s)^{x+y}}\,\mathrm{d}s \;=\; B(x,y).$$

Example: $[0, \frac{\pi}{2}]$ to $(0, 1)$

Consider

$$2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta\ \cos^{2y-1}\theta\,\mathrm{d}\theta.$$

Use the substitution

$$t = \sin^2\theta \qquad (dt = 2\sin\theta\cos\theta\,\mathrm{d}\theta).$$

Then

$$2\sin^{2x-1}\theta\ \cos^{2y-1}\theta\,\mathrm{d}\theta = t^{x-1}(1-t)^{y-1}\,\mathrm{d}t.$$

Hence

$$2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta\ \cos^{2y-1}\theta\,\mathrm{d}\theta = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t = B(x,y).$$

Example: $[0, \frac{\pi}{2}]$ to $(0, 1)$

Consider

$$\int_0^{\frac{\pi}{2}} \sin^x\theta\,\cos^y\theta\,\mathrm{d}\theta.$$

Let

$$u = \sin^2\theta \qquad (\mathrm{d}u = 2\sin\theta\cos\theta\,\mathrm{d}\theta).$$

Then

$$\sin^x\theta = u^{x/2}, \qquad \cos^y\theta = (1-u)^{y/2}.$$

The integral becomes

$$\frac{1}{2}\int_0^1 u^{\frac{x}{2}}(1-u)^{\frac{y}{2}}\,u^{-1/2}(1-u)^{-1/2}\,\mathrm{d}u,$$

which simplifies to

$$\frac{1}{2}B\left(\frac{x+1}{2}, \frac{y+1}{2}\right).$$

Summary of Beta Integrals

The fundamental Beta integral admits three standard forms:

- **Euler's first Beta integral** (on $[0, 1]$):

$$\int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t.$$

- **Euler's second Beta integral** (on $(0, \infty)$):

$$\int_0^\infty \frac{s^{x-1}}{(1+s)^{x+y}}\,\mathrm{d}s.$$

- **Trigonometric Beta integral** (on $[0, \pi/2]$):

$$\int_0^{\pi/2} \sin^{2x-1}\theta\,\cos^{2y-1}\theta\,\mathrm{d}\theta.$$

All evaluate to $B(x, y)$.

Examples of the Three Beta Forms

- **Euler's first form (on $[0,1]$)** Choose $x=2$, $y=3$:

$$\int_0^1 t^1(1-t)^2\,\mathrm{d}t = B(2,3) = \frac{\Gamma(2)\Gamma(3)}{\Gamma(5)} = \frac{1!\,2!}{4!} = \frac{1}{12}.$$

- **Euler's second form (on $(0,\infty)$)** Choose $x=1$, $y=2$:

$$\int_0^\infty \frac{1}{(1+s)^3}\,\mathrm{d}s = B(1,2) = \frac{1}{2}.$$

- **Trigonometric form (on $[0,\pi/2]$)** Choose $x=y=\frac{1}{4}$:

$$\int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{\sin\theta\cos\theta}} = \frac{1}{2}B\left(\tfrac{1}{4},\tfrac{1}{4}\right) = \frac{1}{2\sqrt{\pi}}\Gamma\left(\frac{1}{4}\right)^2.$$

More Beta Integral Examples

- **Euler's first form** ($[0,1]$) Take $x = \frac{5}{2}$, $y = \frac{3}{2}$:

$$I = \int_0^1 t^{3/2}(1-t)^{1/2}\,\mathrm{d}t = B\left(\tfrac{5}{2}, \tfrac{3}{2}\right) = \frac{\Gamma(\frac{5}{2})\Gamma(\frac{3}{2})}{\Gamma(4)}.$$

Using $\Gamma(\frac{5}{2}) = \frac{3\sqrt{\pi}}{4}$ and $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$,

$$I = B\left(\tfrac{5}{2}, \tfrac{3}{2}\right) = \frac{3\pi}{8} \cdot \frac{1}{6} = \frac{\pi}{16}.$$

- **Euler's second form** ($(0,\infty)$) Take $x = \frac{3}{2}$, $y = \frac{5}{2}$:

$$\int_0^\infty \frac{s^{1/2}}{(1+s)^4}\,\mathrm{d}s = B\left(\tfrac{3}{2}, \tfrac{5}{2}\right) = \frac{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})}{\Gamma(4)} = \frac{\pi}{16}.$$

- **Trigonometric form** ($[0,\pi/2]$) Take $x = \frac{3}{2}$, $y = \frac{5}{2}$:

$$\int_0^{\pi/2} \sin^2\theta\,\cos^4\theta\,\mathrm{d}\theta = \frac{1}{2}B\left(\tfrac{3}{2}, \tfrac{5}{2}\right) = \frac{1}{2}\cdot\frac{\pi}{16} = \frac{\pi}{32}.$$

The Shape of the Beta Integrand

Let

$$f(t) = t^{x-1}(1-t)^{y-1}, \qquad 0 < t < 1.$$

Its shape on $(0,1)$ is determined by what happens near the endpoints:

- As $t \to 0^+$:
$$f(t) \sim t^{x-1}.$$
- As $t \to 1^-$:
$$f(t) \sim (1-t)^{y-1}.$$

The parameter x controls the behaviour near 0, and y controls the behaviour near 1.

Poisson's Double Integral

Concept: The Starting Point

Consider

$$I = \iint_{u>0,\ v>0} u^{x-1} v^{y-1} e^{-(u+v)} \, \mathrm{d}u \, \mathrm{d}v, \qquad \Re x > 0,\ \Re y > 0.$$

Proof Sketch: First Evaluation

The integrand factors, and the region is a product:

$$I = \left(\int_0^\infty u^{x-1} e^{-u} \, \mathrm{d}u \right) \left(\int_0^\infty v^{y-1} e^{-v} \, \mathrm{d}v \right) = \Gamma(x)\Gamma(y).$$

Second Evaluation: Change of Variables

Concept: Poisson's Substitution

Set

$$u = rt, \qquad v = r(1-t),$$

with $r > 0$ and $0 < t < 1$. This parametrises the first quadrant.

Proof Sketch

Compute the Jacobian:

$$J = \det \begin{pmatrix} \partial u/\partial r & \partial u/\partial t \\ \partial v/\partial r & \partial v/\partial t \end{pmatrix} = \det \begin{pmatrix} t & r \\ 1-t & -r \end{pmatrix} = -r.$$

Thus

$$|J| = r, \qquad du\,\mathrm{d}v = r\,\mathrm{d}r\,\mathrm{d}t.$$

Second Evaluation: Substitution Yields Beta

Proof Sketch

Substitute into the integrand:

$$u^{x-1} v^{y-1} e^{-(u+v)} = r^{x+y-2} t^{x-1} (1-t)^{y-1} e^{-r}.$$

Multiplying by $du\,\mathrm{d}v = r\,\mathrm{d}r\,\mathrm{d}t$:

$$u^{x-1} v^{y-1} e^{-(u+v)}\,\mathrm{d}u\,\mathrm{d}v = r^{x+y-1} e^{-r}\, t^{x-1} (1-t)^{y-1}\,\mathrm{d}r\,\mathrm{d}t.$$

Thus

$$I = \left(\int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t\right)\left(\int_0^\infty r^{x+y-1} e^{-r}\,\mathrm{d}r\right) = B(x,y)\,\Gamma(x+y).$$

Conclusion of Poisson's Proof

Concept: Equating the Two Evaluations

We found

$$I = \Gamma(x)\Gamma(y) \quad \text{and} \quad I = B(x,y)\Gamma(x+y).$$

Therefore,

$$B(x,y)\Gamma(x+y) = \Gamma(x)\Gamma(y) \quad \Longrightarrow \quad \boxed{B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}}$$

This double-integral argument is the classical proof attributed to Poisson.

A Special Case: $B(x, 1-x)$

Using the Beta–Gamma relation,

$$B(x, 1-x) = \frac{\Gamma(x)\,\Gamma(1-x)}{\Gamma(1)}.$$

Euler's reflection formula states:

$$\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}.$$

Therefore,

$$B(x, 1-x) = \frac{\pi}{\sin(\pi x)}.$$

This identity gives closed forms for many Beta integrals involving complementary parameters.

Dedekind's Proof of Euler's Reflection Formula

Concept 1: Dedekind's Auxiliary Function

$$\phi(x) = \int_0^\infty \frac{t^{x-1}}{1+t}\,\mathrm{d}t, \qquad 0 < x < 1.$$

Concept 2: Two Basic Identities

For every $s > 0$,

$$\int_0^\infty \frac{t^{x-1}}{st+1}\,\mathrm{d}t = \phi(x)s^{-x}, \qquad \int_0^\infty \frac{t^{x-1}}{t+s}\,\mathrm{d}t = \phi(x)s^{x-1}.$$

Hence

$$\phi(x)\,\frac{s^{x-1} - s^{-x}}{s-1} = \int_0^\infty \frac{t^{x-1}(t-1)}{(st+1)(t+s)}\,\mathrm{d}t.$$

Dedekind's Proof: Core Identities

Concept 3: Squaring ϕ

Changing order of integration gives

$$[\phi(x)]^2 = \int_0^\infty \frac{t^{x-1}\log t}{t-1}\,\mathrm{d}t.$$

Concept 4: Symmetry in x

For $0 < y < 1$, integrate in x from $1 - y$ to y:

$$\int_{1-y}^{y} [\phi(x)]^2\,\mathrm{d}x = \int_0^\infty \frac{t^{y-1} - t^{-y}}{t-1}\,\mathrm{d}t.$$

Integrate the last integral in C. 2 from $s = 0$ to ∞ and use C. 3 to obtain

$$\phi(x)\int_{1-x}^{x} [\phi(t)]^2\,\mathrm{d}t = 2\,\phi'(x), \qquad 0 < x < 1.$$

Conclusion of Dedekind's Proof I

Concept 5: Central Symmetry

From $\phi(x) = \phi(1-x)$,

$$\phi'\left(\tfrac{1}{2}\right) = 0, \quad \& \quad \int_{1-x}^{x} [\phi(t)]^2 \,\mathrm{d}t = 2\int_{1/2}^{x} [\phi(t)]^2 \,\mathrm{d}t.$$

Thus

$$\phi(x)\int_{1/2}^{x} [\phi(t)]^2 \,\mathrm{d}t = \phi'(x).$$

Concept 6: Dedekind's Differential Equation

Differentiating

$$\phi(x)\int_{1/2}^{x} [\phi(t)]^2 \mathrm{d}t = \phi'(x)$$

gives the ODE $\phi\,\phi'' - (\phi')^2 = \phi^4$.

Conclusion of Dedekind's Proof II

Concept 7: Solving the ODE

With

$$\phi\left(\frac{1}{2}\right) = \pi, \qquad \phi'\left(\frac{1}{2}\right) = 0,$$

the unique solution is

$$\phi(x) = \pi \csc(\pi x).$$

Concept 8: Euler's Reflection Formula

Since $\phi(x) = \Gamma(x)\Gamma(1-x)$,

$$\boxed{\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}}$$

Homework for Monday

You may use any method you prefer. If you can reduce the integrals to a Beta form or make use of Euler's reflection formula, even better.

Solve the following integrals

1. $$\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1+\cos^2 x}}\,\mathrm{d}x$$

2. $$\int_0^{\infty} \sin(x^2)\,\mathrm{d}x$$

Bring your solution to Wednesday's session.

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$$

to be continued...