Elementary Special Functions & Integrals via ODE Conversions

ODE–Integration Bee Seminar Series

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What Are Special Functions and ODEs?

- **Ordinary differential equations (ODEs):** equations that connect a function and its derivatives.

- **Special functions:** functions that arise repeatedly in mathematics and physics, often discovered while solving differential equations or evaluating fundamental integrals.

  Some have descriptive names (like the Gamma function Γ, the polygamma function ψ_s, the Beta function B, or the polylogarithm Li_s),

  while others are named after people (like the Bessel function J_ν, the Legendre polynomial P_n, the Hermite polynomial H_n, or the Airy function Ai).

- **Special values:** evaluations at notable inputs (often integers or fractions) that yield important constants — e.g. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \mathrm{Li}_2(1) = \frac{\pi^2}{6}$.

Plan for the Mini-Course

- **Gamma Function** $\Gamma(x)$: a smooth extension of factorials $(n-1)!$ to all real $x > 0$.
- **Beta Function** $B(x, y)$: an integral on $[0, 1]$ with the identity

$$B(x, y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}.$$

- **Elementary ODEs:** constant–coefficient ODEs that reduce to polynomials.
- **Conversion Trick:** integral $\rightarrow$ ODE $\rightarrow$ solve ODE $\rightarrow$ value of the integral.

Why Special Functions?

Why Special Functions?

- Many non-elementary integrals lead naturally to special functions, e.g.

$$\int_0^\infty e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2} = \frac{1}{2}\Gamma\left(\frac{1}{2}\right).$$

- They arise as solutions of classical differential equations, e.g.

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \text{ has solution } y = J_\nu(x).$$

- Familiarity with them reveals structure in complex expressions, e.g.

$$\sum_{n=0}^{\infty} \frac{(-z)^n}{(n!)^2} = J_0(2\sqrt{z}).$$

Why Special Functions?

From a Complicated Integral to a Series

They allow us to express complex results more simply. For instance,

$$\int_0^\infty x^{\nu-1} e^{-\beta x-\gamma/x}\,\mathrm{d}x = \left(\frac{\gamma}{\beta}\right)^{\nu/2} \frac{\pi}{\sin(\pi\nu)}[A_{-\nu}(z) - A_\nu(z)],$$

where

$$A_{-\nu}(z) = \left(\frac{z}{2}\right)^{-\nu} \sum_{k=0}^{\infty} \frac{(z^2/4)^k}{k!\,\Gamma(1-\nu+k)}, \qquad A_\nu(z) = \left(\frac{z}{2}\right)^{\nu} \sum_{k=0}^{\infty} \frac{(z^2/4)^k}{k!\,\Gamma(1+\nu+k)},$$

with

$$z = 2\sqrt{\beta\gamma}, \qquad \Re\beta, \Re\gamma > 0, \qquad \nu \notin \mathbb{Z}.$$

Connection to the Bessel Function

Simplified Expression

With the modified Bessel function K_ν, we have

$$\int_0^\infty x^{\nu-1} e^{-\beta x-\gamma/x}\,\mathrm{d}x = 2\left(\frac{\gamma}{\beta}\right)^{\nu/2} K_\nu(2\sqrt{\beta\gamma}),$$

valid for $\Re\beta, \Re\gamma > 0,\ \nu \notin \mathbb{Z}$.

(The lengthy double series collapses beautifully into a single special function.)

The Gamma Function

Extending the factorials

- Factorials: $n! = 1 \cdot 2 \cdot 3 \cdots n$
- Only defined for whole numbers
- What is $\frac{1}{2}!$ or $\frac{3}{4}!$?
- Goal: a **smooth extension** of factorials to all real (and later complex) numbers

What is a Smooth Extension?

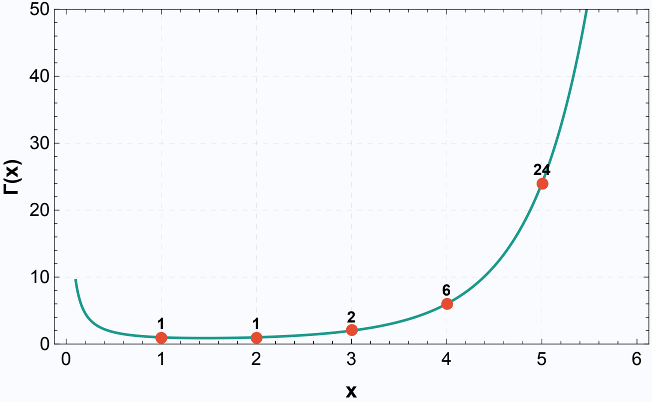

Figure: Plot of $\Gamma(x)$ passing smoothly through factorial values

An Integral That Creates a Function

A curious example

$$I(x) = \int_0^\infty t^{x-1} e^{-t} \, \mathrm{d}t$$

- This integral converges for every $x > 0$.
- For integer x, it gives:

$$I(1) = 1, \quad I(2) = 1!, \quad I(3) = 2!, \quad I(4) = 3!, \ldots$$

- We name this function $\Gamma(x)$ — it **extends factorials continuously**.
- So, a definite integral produced a new function — our first **special function**.

How the Gamma Function Extends Factorials

Proof Sketch: Setup

Recall

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,\mathrm{d}t, \qquad x > 0.$$

For $x > 0$, consider

$$\Gamma(x+1) = \int_0^\infty t^{x} e^{-t}\,\mathrm{d}t.$$

We apply integration by parts with $u = t^x$, $\mathrm{d}v = e^{-t}\mathrm{d}t$, so that $\mathrm{d}u = x\,t^{x-1}\mathrm{d}t$ and $v = -e^{-t}$.

Substituting,

$$\Gamma(x+1) = -t^x e^{-t}\Big|_0^\infty + x\int_0^\infty t^{x-1} e^{-t}\,\mathrm{d}t.$$

Deriving the Recurrence $\Gamma(x+1) = x\,\Gamma(x)$

Proof Sketch: Simplification

The boundary term vanishes because $t^x e^{-t} \to 0$ as $t \to \infty$ and at $t = 0$. Hence

$$\Gamma(x+1) = x\,\Gamma(x).$$

For integers $n \geq 1$,

$$\Gamma(n+1) = n\,\Gamma(n) = n!,$$

so the Gamma function truly *extends the factorials*.

(This simple identity is the heartbeat of the entire Γ family.)

The Gaussian Integral and the Gamma Function

Example: Gaussian Integral

$$f(a) = \int_0^\infty e^{-ax^2}\,\mathrm{d}x, \qquad a > 0.$$

This integral appears everywhere — in probability, statistics, and physics.

It is also closely related to the **Gamma function**:

$$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t}\,\mathrm{d}t.$$

(Let's see how a simple substitution turns one integral into the other.)

From $f(a)$ to the Gamma Function

A simple substitution

To see the connection, let

$$t = ax^2 \quad \Longrightarrow \quad \mathrm{d}t = 2ax\,\mathrm{d}x,\; x = \frac{1}{2\sqrt{a}}t^{-\frac{1}{2}}.$$

Substituting into $f(a)$ gives

$$f(a) = \frac{1}{2\sqrt{a}}\int_0^\infty t^{-1/2}e^{-t}\,\mathrm{d}t.$$

The remaining integral is just the Gamma function at $p = \frac{1}{2}$:

$$f(a) = \frac{1}{2\sqrt{a}}\Gamma\left(\tfrac{1}{2}\right).$$

Introducing a Parameter

Concept: A new way to look at $f(a)$

Think of a as a variable, and consider

$$I(s) = \int_0^\infty f(a)\, e^{-sa}\, \mathrm{d}a = \int_0^\infty \int_0^\infty e^{-a(x^2+s)}\, \mathrm{d}x\, \mathrm{d}a, \qquad s > 0.$$

Because the integrand is positive and decays quickly, we can **swap the order of integration**:

$$I(s) = \int_0^\infty \int_0^\infty e^{-a(x^2+s)}\, \mathrm{d}a\, \mathrm{d}x.$$

The inner integral is easy:

$$\int_0^\infty e^{-a(x^2+s)}\, \mathrm{d}a = \frac{1}{x^2+s}.$$

Introducing a Parameter (II)

Evaluating $I(s)$

We now have

$$I(s) = \int_0^\infty \frac{1}{x^2 + s} \, \mathrm{d}x.$$

Using the substitution $x = \sqrt{s} \tan \theta$,

$$I(s) = \frac{1}{\sqrt{s}} \int_0^{\pi/2} \mathrm{d}\theta = \frac{\pi}{2\sqrt{s}}.$$

Hence

$$I(s) = \int_0^\infty f(a) e^{-sa} \, \mathrm{d}a = \frac{\pi}{2\sqrt{s}}.$$

(This relation tells us how $f(a)$ must depend on a.)

Connecting Back to the Gamma Function

Concept: Relating $I(s)$ and $\Gamma(p)$

Recall the Gamma function:

$$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t}\,\mathrm{d}t.$$

Substituting $t = sa$ gives

$$\Gamma(p) = s^p \int_0^\infty a^{p-1} e^{-as}\,\mathrm{d}a.$$

Hence,

$$\int_0^\infty a^{p-1} e^{-as}\,\mathrm{d}a = \frac{\Gamma(p)}{s^p}.$$

For $p = \frac{1}{2}$,

$$\int_0^\infty a^{-1/2} e^{-as}\,\mathrm{d}a = \frac{\Gamma(\frac{1}{2})}{\sqrt{s}}.$$

Connecting Back to the Gamma Function (II)

Concept: Determining $f(a)$

From the parameter method, we know

$$I(s) = \frac{\pi}{2\sqrt{s}} = \int_0^\infty \frac{\pi\, a^{-1/2}}{2\,\Gamma(\frac{1}{2})} e^{-sa}\,\mathrm{d}a.$$

This means that

$$f(a) = \frac{\pi\, a^{-1/2}}{2\,\Gamma(\frac{1}{2})}.$$

But earlier, we found directly that

$$f(a) = \frac{1}{2\sqrt{a}}\,\Gamma\left(\frac{1}{2}\right).$$

Evaluating $\Gamma\left(\frac{1}{2}\right)$

Concept: Comparing the two forms of $f(a)$

Equating both expressions for $f(a)$, we have

$$\frac{1}{2\sqrt{a}}\,\Gamma\left(\tfrac{1}{2}\right) = \frac{\pi\, a^{-1/2}}{2\,\Gamma\left(\tfrac{1}{2}\right)}.$$

Multiplying both sides by $2\sqrt{a}\,\Gamma\left(\tfrac{1}{2}\right)$ gives

$$\Gamma^2\left(\tfrac{1}{2}\right) = \pi.$$

Therefore,

$$\boxed{\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}}$$

What Is an Ordinary Differential Equation?

Concept: A first encounter

- An **ordinary differential equation (ODE)** connects a function $y(x)$ with its *derivatives*.

$$y'(x),\ y''(x),\ y^{(3)}(x),\ \ldots$$

- It is called *ordinary* because the function depends on only *one variable* (unlike partial differential equations, PDEs).
- Example:

$$y'(x) = 2y(x) \quad \Longrightarrow \quad \text{solution } y(x) = Ce^{2x}.$$

- Many physical laws (cooling, motion, growth, decay) take this form.

First–Order ODEs and Separation of Variables

Concept: A gentle start before higher–order equations

The simplest differential equations involve only the first derivative:

$$\frac{dy}{dx} = g(x)\,h(y).$$

If we can separate x–terms and y–terms, the equation is **separable**:

$$\frac{dy}{h(y)} = g(x)\,\mathrm{d}x.$$

Integrating both sides gives the solution:

$$\int \frac{dy}{h(y)} = \int g(x)\,\mathrm{d}x + C.$$

Example

Concept: An exponential growth equation

Consider

$$y' = ky.$$

It is separable:

$$\frac{dy}{y} = k\,\mathrm{d}x.$$

Integrating both sides gives

$$\ln y = kx + C \quad \Longrightarrow \quad y = Ce^{kx}.$$

Elementary ODEs with Constant Coefficients (I)

ODE Recipe: From calculus to algebra

- A **constant–coefficient ODE** has fixed numbers (not functions of x) multiplying the derivatives.

$$a_2 y'' + a_1 y' + a_0 y = 0.$$

- We make the standard **exponential ansatz** $y = e^{rx}$. Substituting gives

$$(a_2 r^2 + a_1 r + a_0) e^{rx} = 0.$$

  Since $e^{rx} \neq 0$,

$$a_2 r^2 + a_1 r + a_0 = 0,$$

  so we end up with a simple **polynomial equation** in r.

Elementary ODEs with Constant Coefficients (II)

ODE Recipe: From calculus to algebra

- This is called the **characteristic (or auxiliary) equation**.
- So constant–coefficient ODEs reduce to solving a polynomial!

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}.$$

- These are our most "elementary" ODEs.

Shapes of the Solutions (I)

Concept: Real roots of the characteristic equation

For

$$a_2 y'' + a_1 y' + a_0 y = 0 \quad \Longrightarrow \quad a_2 r^2 + a_1 r + a_0 = 0,$$

each root r gives an exponential e^{rx}.

1. Two distinct real roots r_1, r_2:

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}.$$

Example: $y'' - 3y' + 2y = 0 \implies r = 1, 2 \implies y = C_1 e^{x} + C_2 e^{2x}$.

2. Repeated real root r:

$$y = (C_1 + C_2 x) e^{rx}.$$

Example: $y'' - 2y' + y = 0 \implies r = 1 \implies y = (C_1 + C_2 x) e^{x}$.

Shapes of the Solutions (II)

Concept: Complex conjugate roots

If $r = \alpha \pm i\beta$ with real α, β, the solution combines exponentials and oscillations:

$$y = e^{\alpha x}\big(C_1 \cos \beta x + C_2 \sin \beta x\big).$$

Example: $y'' + y = 0 \Rightarrow r = \pm i$. Hence

$$y = C_1 \cos x + C_2 \sin x.$$

Here $\alpha = 0, \beta = 1$: the solution neither grows nor decays—it oscillates.

(Imaginary roots $\implies$ waves and harmonic motion.)

Idea: Treat a Parameter as a Variable

Concept: From integrals to ODEs

- Many integrals depend on a parameter, e.g.

$$I(\alpha) = \int_0^\infty e^{-x^2} \cos(\alpha x)\, \mathrm{d}x, \qquad \alpha \in \mathbb{R}.$$

- Instead of attacking the integral directly, we:
    - view $I(\alpha)$ as an unknown function of α;
    - differentiate $I(\alpha)$ with respect to α under the integral sign;
    - simplify until we obtain an $\boxed{\text{ODE for } I(\alpha)}$.
- Then we solve the ODE and finally fix an integration constant with a known value of the integral.

A Bee-Style Question

Example: Gaussian-damped integral

Evaluate

$$\int_0^\infty e^{-2x^2} \cos(3x)\,\mathrm{d}x$$

Instead of attacking it head-on, we study

$$F(a,b) := \int_0^\infty e^{-ax^2} \cos(bx)\,\mathrm{d}x, \qquad a > 0,\ b \in \mathbb{R}.$$

Our goal: understand $F(a,b)$ in general, and then plug in $a = 2,\ b = 3$.

How Does the Graph Look?

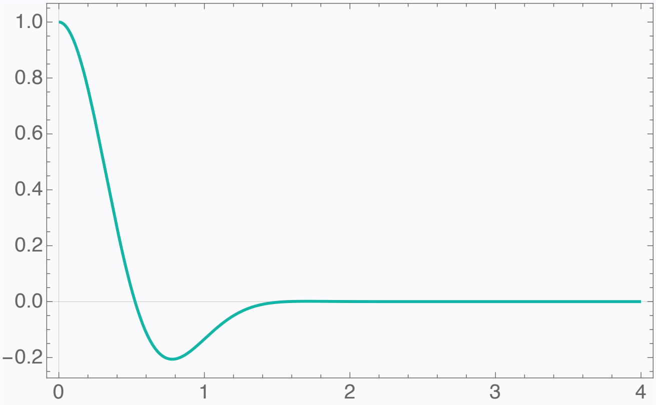

Figure: Plot of $e^{-2x^2}\cos(3x)$ on $(0,\infty)$: a decaying oscillation

From the Integral to an ODE (I)

ODE Recipe: Differentiating under the integral sign

Fix $a > 0$ and view $F(a, b)$ as a function of b:

$$F(a, b) = \int_0^\infty e^{-ax^2} \cos(bx) \, \mathrm{d}x.$$

(Gaussian decay allows differentiation inside the integral.)

- Differentiate with respect to b:

$$\frac{\partial F}{\partial b} = -\int_0^\infty x \, e^{-ax^2} \sin(bx) \, \mathrm{d}x.$$

- Use the identity

$$xe^{-ax^2} = -\frac{1}{2a} \frac{\mathrm{d}}{\mathrm{d}x} \left(e^{-ax^2} \right).$$

From the Integral to an ODE (II)

ODE Recipe: Preparing for integration by parts

Substituting

$$xe^{-ax^2} = -\frac{1}{2a}\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-ax^2}\right)$$

into the derivative gives

$$\frac{\partial F}{\partial b} = \frac{1}{2a}\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-ax^2}\right)\sin(bx)\,\mathrm{d}x.$$

We now integrate this expression by parts with respect to x.

Deriving the ODE

Proof Sketch: Integration by parts

Integrate by parts:

$$\int_0^{\infty} \mathrm{d}\left(e^{-ax^2}\right)\sin(bx) = \left[e^{-ax^2}\sin(bx)\right]_0^{\infty} - b\int_0^{\infty} e^{-ax^2}\cos(bx)\,\mathrm{d}x.$$

The boundary term vanishes, so

$$\int_0^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-ax^2}\right)\sin(bx)\,\mathrm{d}x = -b\,F(a,b).$$

Therefore

$$\boxed{\frac{\partial F}{\partial b} = \frac{1}{2a}\left(-bF(a,b)\right) = -\frac{b}{2a}F(a,b).}$$

Solving the ODE for $F(a, b)$

Concept: Shape of $F(a, b)$

From

$$\frac{\partial F}{\partial b} = -\frac{b}{2a} F,$$

we separate variables:

$$\frac{1}{F}\frac{\partial F}{\partial b} = -\frac{b}{2a} \quad \Longrightarrow \quad \ln F(a, b) = -\frac{b^2}{4a} + C(a).$$

Exponentiating,

$$F(a, b) = A(a) \exp\left(-\frac{b^2}{4a}\right), \qquad a > 0,$$

where $A(a)$ is a constant with respect to b.

Evaluating the Integral

Fixing $A(a)$

At $b = 0$,

$$F(a,0) = \int_0^\infty e^{-ax^2}\,\mathrm{d}x = \frac{1}{\sqrt{a}} \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right),$$

so $A(a) = \dfrac{\sqrt{\pi}}{2\sqrt{a}}$, and

$$F(a,b) = \frac{\sqrt{\pi}}{2\sqrt{a}} \exp\left(-\frac{b^2}{4a}\right).$$

Back to the Bee problem

In particular,

$$\int_0^\infty e^{-2x^2}\cos(3x)\,\mathrm{d}x = F(2,3) = \frac{\sqrt{\pi}}{2^{\frac{3}{2}}} \exp\left(-\frac{3^2}{2^3}\right).$$

“

Special functions are useful and those who need them and those who know them should start to talk to each other... The mathematical community at large needs the education on the usefulness of special functions more than most other people who could use them.

— Richard Askey

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$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

to be continued...